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On the displacement height in the logarithmic velocity profile

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The displacement height appears in the logarithmic velocity profile for rough-wall boundary layers as a reference height for the vertical co-ordinate. It is shown that this height should be regarded as the level at which the mean drag on the surface appears to act. The equations of motion then show that this also coincides with the average displacement thickness for the shear stress.

A simple analytical model, experimental results and dimensional analysis are all used to indicate how the displacement height depends upon the detailed geometry of the roughness elements.

1. Introduction

The engineering and meteorological literature relating to turbulent flow over rough surfaces contains a great deal of confusion about the displacement height d, which appears in the customary logarithmic law for the velocity profile;

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\frac{z-d}{z_0}.$$

For common types of surface cover the roughness length z_0 is observed to increase with the heights of the roughness elements, so it is regarded as the basic measure of the degree of surface roughness. In using a surface-friction law to estimate shear stress it is true that z_0 is by far the more significant parameter and that d may be neglected or approximated accordingly. However this is not permissible if one is trying to determine z_0 for a particular surface. This can only be done by fitting the log-law to a measured velocity profile, and in atmospheric measurements especially one typically does not know $u_*/\kappa, d$ or z_0 so all three have to be found from a velocity profile measured at perhaps five or six heights. In order to reduce the standard error of the results d is often given an assumed value (usually zero). However, Oke (1974) points out that one effect of doing this may be just to increase the scatter in z_0 (and perhaps the von Kármán constant κ) obtained by different investigators for the same type of roughness. Our aim here is to establish a physical meaning for d and to investigate how it depends on the variables describing the geometry and layout of a rough surface, that it may be better estimated in practical situations.

The most obvious definition of displacement height is that it is the average surface elevation (sometimes called the geometric height); that is, the level which would be obtained by flattening out all the roughnesses into a smooth surface. An extension to this idea is that all regions of separated flow should be considered as part of the

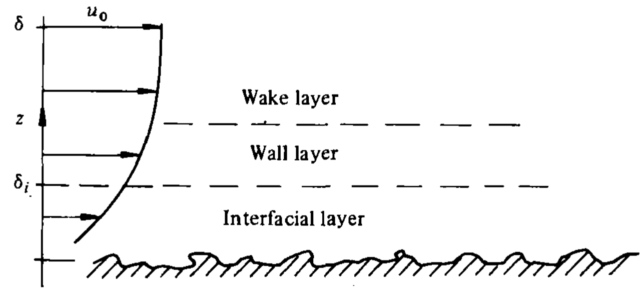

FIGURE 1. Regions of boundary-layer flow (not to scale).

roughness elements before the levelling process is carried out. Sayre & Albertson (1963) have considered both definitions, and show that neither leads to a consistent experimental value for the von Kármán constant.

More recently Thom (1971) found experimentally that the displacement height of a dense needle-like roughness was 109 ± 2 mm and that the centre of moment of the forces acting on the elements was 108 ± 3 mm. This remarkably close coincidence led him to suppose that for all types of roughness the displacement height is 'the level of the actual mean momentum sink'. It is shown below that this definition of d is implicit in the way the logarithmic law is derived, although it is not usually stated. It is then shown that this definition also leads to a new interpretation for the displacement height in zero pressure gradient flows.

2. Logarithmic law

The logarithmic profile for the mean velocity now has so much experimental and theoretical support that it is usually taken for granted. However, since the zero-plane displacement height is of particular interest here, it is necessary to reconsider the logarithmic expression in which it appears, paying particular attention to the assumptions used to derive it with the familiar arguments of similarity and dimensional analysis. We shall consider a turbulent boundary layer flowing over a rough surface for which both the boundary-layer thickness and the horizontal scale on which the flow is developing are much greater than any dimension defining the roughness elements. The layer is subdivided into wall and wake regions, and an interfacial layer, as sketched in figure 1.

The height δ_i is defined as that above which the flow is not affected by individual roughness elements. There is then no mean flow across $z = \delta_i$, and in the absence of an external horizontal pressure gradient the mean stress experienced by the base of the wall layer must equal the average horizontal force per unit plan area, τ_0, acting on the surface. If the average moment per unit plan area exerted by these forces is M, then the level of action of τ_0 is a distance M/τ_0 above any arbitrary origin for the vertical co-ordinate z. It is now postulated that the wall layer depends primarily on τ_0 but only weakly on any other property of the flow in the interfacial layer; similarly it cannot distinguish any details of the surface except the level at which this stress appears to act. The appropriate velocity scale and apparent origin for the wall layer are therefore u_* and $z = d$ respectively, where

$$u_* = (\tau_0/\rho)^{\frac{1}{2}}, \quad d = M/\tau_0. \tag{1}$$

If the wall layer is sufficiently remote from the free stream that the free-stream velocity u_0 and overall boundary-layer thickness δ do not affect it directly, then the mean velocity there can be written

$$u = u_* f\left(\frac{z-d}{h}, \frac{hu_*}{\nu}, \frac{\lambda_1}{h}, \frac{\lambda_2}{h}, \ldots\right). \tag{2}$$

Here h is a typical roughness height and the λ_i are other length scales defining the roughness geometry.

In the outer wake region the velocity is nearly equal to the free-stream value u_0. The distance from the wall is comparable with the boundary-layer thickness, δ, and since this is much greater than any roughness dimension it follows that

$$u = u_0 + u_* g\left(\frac{z-d}{\delta}\right). \tag{3}$$

Here the use of u_* and the same displacement height d are required by the next step.

If there is a region where these two layers overlap, then equating the derivatives of (2) and (3) there and multiplying by $(z-d)$ gives

$$\left(\frac{z-d}{h}\right) f' = \left(\frac{z-d}{\delta}\right) g',$$

where the prime indicates differentiation by the variable containing z in each case. By separation of variables both sides are independent of $(z-d)$ and equal to u_*/κ, say, when equation (2) becomes

$$\left.\begin{aligned} \frac{u}{u_*} &= \frac{1}{\kappa}\ln\frac{z-d}{h} + C\left(\frac{hu_*}{\nu}, \frac{\lambda_1}{h}, \frac{\lambda_2}{h}, \ldots\right), \\ \text{or} \qquad \frac{u}{u_*} &= \frac{1}{\kappa}\ln\frac{z-d}{z_0}, \end{aligned}\right\} \tag{4}$$

where z_0/h is a function of hu_*/ν, λ/h, etc. The function g is also logarithmic, and equating (2) and (3) in the overlap region gives

$$\frac{u_0}{u_*} = \frac{1}{\kappa}\ln\frac{\delta}{z_0} + \text{constant}. \tag{5}$$

This derivation of the log-law makes it clear that d has a dynamic significance. Whatever the origin of z, the displacement height d adjusts the reference level for the velocity profile to the height at which the mean surface shear appears to act. Further, in any particular problem the balance of horizontal forces (on a larger scale than that considered here) leads to an estimate for δ. For example, in pipe flow δ corresponds to the pipe radius, in the planetary boundary layer it depends on u_*/f (where f is Coriolis acceleration) and on flat self-preserving boundary layers $\delta/(x)$ is proportional to $x(u_*/u_0)^2$ (Townsend 1976). Therefore, given u_0, the roughness length z_0 is the length scale which expresses the *magnitude* of forces which act on the surface (via equation (5)), whereas d is related to the *distribution* of these forces.

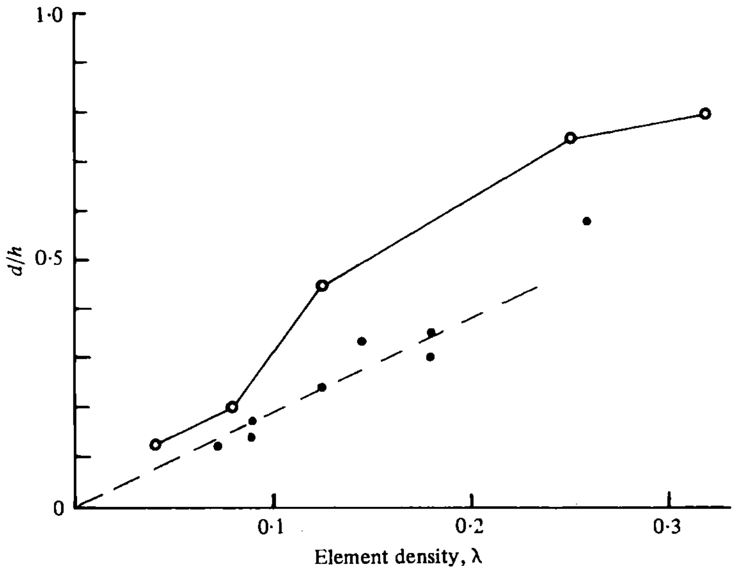

FIGURE 2. d/h versus λ; —●—, Counihan (1971); —○—, Lee & Soliman (1977).

3. Dimensional analysis and drag partition

In this section the relationship between the lengths z_0 and d and the actual geometry of the rough surface will be examined. The derivation above gave

$$z_0/h = F\left(\frac{u_* h}{\nu}, \frac{\lambda_1}{h}, \frac{\lambda_2}{h}, \dots\right). \tag{6}$$

A second expression can be obtained using the definition of d. If D is the drag on a single roughness element the moment on the element due to horizontal forces can be written $c_m hD$, where c_m is the appropriate moment coefficient. If there are n elements in an area S the total moment on the area is $nc_m hD$ whereas the total drag is $\rho u_*^2 S$. Equation (1) then leads to

$$\frac{d}{hc_m} = \frac{nD}{\rho u_*^2 S}, \tag{7}$$

in which the right-hand side can be recognized as the proportion of total drag carried by the upstanding roughness elements. This drag partition will be a function of element height, shape and spacing, so that d/h can be expressed as a function of these variables analogous to equation (6). Thus although z_0 and d are used to describe the parts of the flow which do not feel the effects of individual roughness elements, each is nevertheless influenced by the details of the surface geometry.

Further progress can be made with the aid of postulates about the drag partition. First suppose that the roughness elements are so widely spaced that the velocity profile incident on each element is virtually independent of the element density. Then the drag on an element of frontal area A_f could be written

$$D = \rho u_*^2 c_D A_f$$

and so

$$d/h = c_D c_m \lambda, \quad \text{where} \quad \lambda = nA_f/S. \tag{8}$$

Roughness type	h	d/h	Source
Cubes	0·34–2·4 mm	0·73	O'Loughlin & Annambhotla (1969)
Sand roughness	2 mm	0·72	Blihco & Partheniades (1971)
Sand roughness	2–9 mm	0·65–0·69	Grass (1971)
Sand roughness	0·5–38 mm	0·71	Kamphius (1974)
Densely packed rods	0·14 m	0·77	Thom (1971)
Crops and forests	1–8 m	0·54–0·84	Kondo (1971)
Grass → forest	0·02–20 m	0·64	Stanhill (1969)

TABLE 1. Measurements of d/h for common roughness types.

For this very sparse array the drag and moment coefficients will be independent of the element spacing λ, whereupon for a given type of roughness d/h is linearly dependent on λ. Counihan (1971) and Lee & Soliman (1977) studied the effect of λ on the d/h inferred by fitting equation (4) to measured velocity profiles. Their results are consistent with the predicted linear relationship, as shown in figure 2, even at values of λ which are much too high to satisfy the assumptions leading to equation (8).

In the other extreme of closely packed elements there will be little friction drag on the intervening surface and so the drag partition will approach unity. In that case equation (7) gives $d/h = c_m$. This result has an immediate application in that the moment coefficient (which is relatively difficult to measure) can be estimated by measuring the displacement height of the velocity profile. There are numerous measurements of d/h available for various roughnesses, but there appears to be none with the corresponding data for c_m. However, in Thom's (1971) experiment the drag was clearly all carried by the needle-like roughness elements, so Thom was able to calculate the level of action of the drag force (though he did not measure this directly) and show that it did coincide with the displacement height of the velocity profile above the roughness.

Table 1 collects together other values of d/h from measurements in pipe flow, agricultural meteorology, hydraulics and wind engineering. Over an exceptionally large range of roughness the approximation $d/h = 0{\cdot}7$ is remarkably good. The explanation for this must be that the rough surfaces which are commonly encountered have a similar density of roughness elements (λ). Counihan (1971) and Lee & Soliman (1977) made wind-tunnel experiments in which λ was varied over a wide range, with the expected result that d/h increases as the surface density of roughnesses increases whereas z_0/h first increases and then decreases.

Perry, Schofield & Joubert (1969) also made some important measurements on two-dimensional roughness elements in which λ was varied. They chose to use the Clauser rough-wall profile referenced to a distance ϵ below the roughness tops,

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\left[\frac{(z-h+\epsilon)u_*}{\nu}\right] + A - \frac{\Delta u}{u_*},$$

and found that
$$\frac{\Delta u}{u_*} = \frac{1}{\kappa}\ln\frac{\epsilon u_*}{\nu} + B,$$

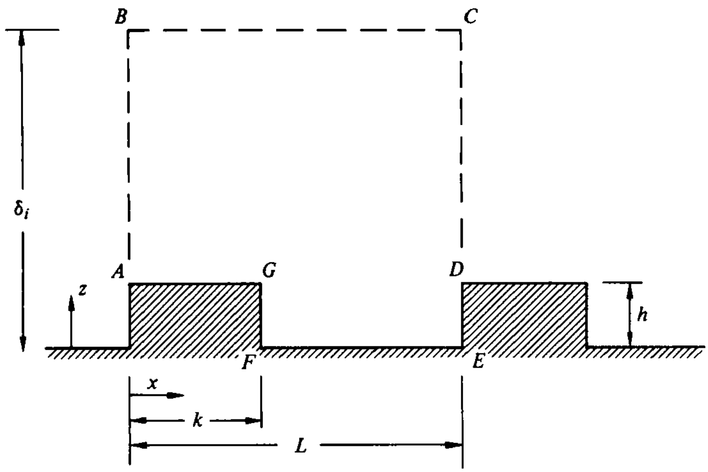

FIGURE 3. An idealized rough surface.

where A and B are constants. Comparing these results with equation (4) leads to the result $\epsilon = h-d$, and also that

$$\frac{\epsilon}{z_0} = \frac{h-d}{z_0} = \text{constant}$$

for this type of roughness. Now for many common rough surfaces it is found that ϵ/h is a constant – Perry *et al.* called these 'k'-type roughnesses, for which it follows that z_0/h and z_0/d are also constants. They also distinguished a different 'd'-type roughness in which the elements are so close together that the flow effectively skims across the gaps. For these Perry *et al.* found $B = -0{\cdot}4$ and $A = 5{\cdot}1$, which leads to $d = h - 9z_0$. Thus as the elements are crowded closer and closer together so that the apparent roughness (z_0) decreases, the displacement height will approach the level of the tops of the roughnesses as expected.

4. The displacement thickness for total shear stress

The displacement height d has been interpreted as the height at which the average drag on the surface appears to act. Using the equations of motion, it is not difficult to calculate the moment and drag on the surface in terms of the mean fluid properties. Equation (1) then leads to yet another interpretation for the displacement height.

We consider the flow in the interfacial layer over an idealized rough surface as shown in figure 3. It is assumed that there is no difference between fluid variables at sections AB and DC. The roughness is assumed to be two-dimensional for simplicity, but the same results are obtained for more general shapes.

The appropriate equations are those of horizontal momentum and continuity averaged over time and over the third direction;

$$\rho\frac{\partial(u^2)}{\partial x} + \rho\frac{\partial(uw)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{12}}{\partial z}, \tag{9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{10}$$

Here the stresses T_{11} and T_{12} include the Reynolds stresses, so u and w are the mean velocity components. If equation (9) is integrated over x, then multiplied by z and integrated over z, we obtain for the rectangle $DABCD$:

$$\int_h^{\delta_i} z[\rho u^2 + p - T_{11}]_{x=0}^{L}\,dz = \int_0^L [zT_{12} - z\rho uw]_{z=h}^{\delta_i}\,dx - \int_0^L\int_h^{\delta_i} [T_{12} - \rho uw]\,dz\,dx; \quad (11)$$

and for $DEFGD$, since u and T_{11} vanish on a solid vertical boundary,

$$\int_0^h z[p]_{x=k}^{L}\,dz = \int_k^L [zT_{12} - z\rho uw]^{z=h}\,dx - \int_k^L\int_0^h [T_{12} - \rho uw]\,dz\,dx. \quad (12)$$

Since there are no differences between flow variables at sections AB and CD the left-hand side of equation (11) is zero. Adding to (12) gives

$$M = \int_0^L \delta_i[T_{12} - \rho uw]^{z=\delta_i}\,dx - \iint_{\text{fluid}} [T_{12} - \rho uw]\,dx\,dz,$$

where M is the moment on the surface due to *horizontal* forces only and the double integral is taken over the fluid contained in the region $ABCDEFGA$. Noting that

$$\iint_{\text{fluid}} dx\,dz = \int_0^L \delta_i\,dx - kh,$$

we can write the above equation as

$$M = \iint_{\text{fluid}} ([T_{12} - \rho uw]^{z=\delta_i} - [T_{12} - \rho uw])\,dx\,dz + kh[T_{12} - \rho uw]^{z=\delta_i}, \quad (13)$$

where we have taken $[T_{12} - \rho uw]^{z=\delta_i} = \tau_0$ to be independent of x in accordance with the requirement that individual roughnesses do not affect the flow at this level. The total horizontal force acting on the surface is the same as that applied to BC;

$$D = L\tau_0. \quad (14)$$

This force appears to act at a height of d given by equation (1); using (13) and (14)

$$d - \bar{h} = \frac{1}{L\tau_0}\iint_{\text{fluid}} (\tau_0 - [T_{12} - \rho uw])\,dx\,dz, \quad (15)$$

where $\bar{h}$ is the average elevation of the surface, $\bar{h} = kh/L$.

The right-hand side of this equation is seen to be the average displacement thickness of the total stress $T_{12} - \rho uw$, as follows. This stress gradually decreases with height as the momentum of the flow is extracted by the drag forces on the roughness elements. At an arbitrary section a displacement thickness for this stress may therefore be constructed as shown in figure 4. The local average value of this thickness is then given by the operation $L^{-1}\int_0^L [\,]\,dx$, the final result being the right-hand side of equation (15). The appearance of the term $-\rho uw$ in this expression means that the so-called 'wave-induced' stresses are included as part of the total shear stress. These stresses arise from spatial averaging of flows which are periodic in space (as here), so they are analogous to the Reynolds stresses which result from time averaging.

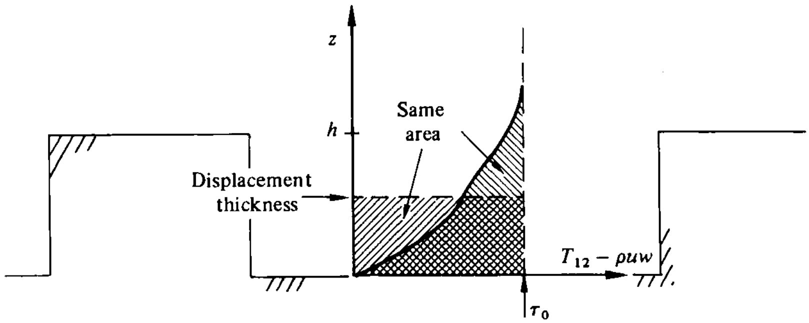

FIGURE 4. Typical shear-stress profile and corresponding displacement thickness.

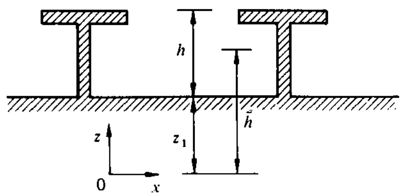

FIGURE 5. Reference levels for a rough surface.

If three-dimensional roughnesses are considered the same results are obtained. The definition (1) for d is still valid. The length $\bar{h}$ in (15) is still the average surface elevation (as defined in § 1), and the integral in (15) is made over a fluid volume with L replaced by the corresponding plan area so the overall expression is again the average displacement thickness for the stress. However, if the condition that no boundary-layer development takes place on the length scale L is relaxed, or overall horizontal pressure gradients are included, the expression corresponding to (15) becomes much more complicated and has no obvious physical significance.

To illustrate the implications of equation (15), we consider the roughnesses shown in figure 5. The reference level for the surface is not defined initially – the top of the elements would be no more an obvious choice than their base – so we have used an arbitrary origin, O. The first result of (15) is then that this reference must be adjusted to the average elevation of the roughnesses. If anything it is this elevation which is the intuitive choice of reference, and it is gratifying to have this confirmed. But this height is then adjusted again by an amount corresponding to the stress displacement thickness, which we denote by Δ. Now, if the elements of roughness are widely spaced then the stress would be constant over most of the fluid and thus Δ would be small (relative to h). Similarly, $\bar{h}$ would be only slightly greater than z_1, so the overall d would be slightly above the base of the roughnesses. At the opposite extreme where the roughnesses are so dense that their tops are actually touching, the shear stress in the fluid trapped beneath would be zero. According to (15) the displacement height is then $z_1 + h$ as we expect, and moreover, if the stagnant fluid were replaced by solid material, exactly the same result for d would appear.

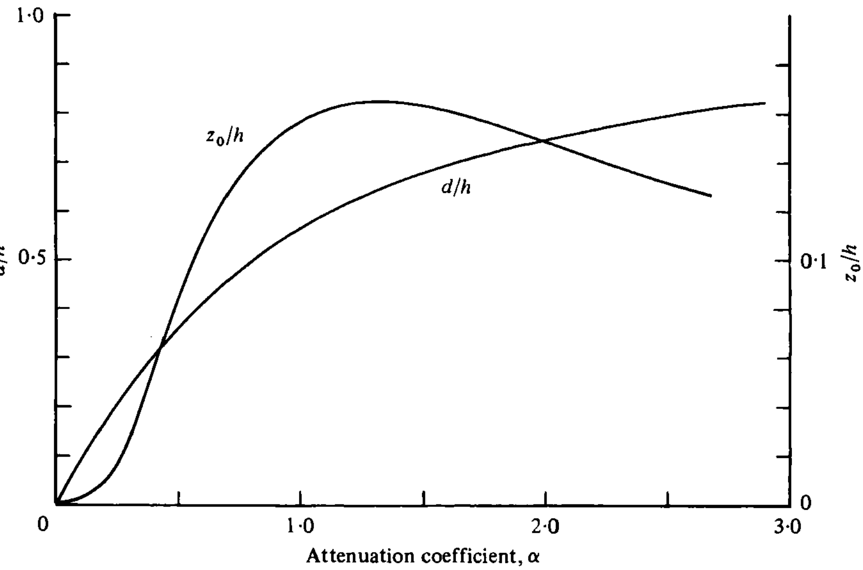

FIGURE 6. Calculated d/h and z_0/h for rod-like roughness elements.

5. A derivation of d for rod-like roughness

In order to use equation (15) to predict d, it is clear that the displacement thickness for the stress must first be found. In many cases this will be extremely difficult, if not impossible, so this will not be a sensible way of finding d unless the nature of the flow around the roughness elements is particularly well understood. This is the case for long rod-like roughnesses, for which an expression for the displacement height is obtained below. Although the results are reasonable this solution has a number of deficiencies, so what follows should be viewed primarily as an illustration of the implications of equation (15).

For long rod-like roughnesses the momentum equation can be written

$$\frac{dT_{12}}{dz} = \frac{\rho\lambda c_D u^2(z)}{2h}, \tag{16}$$

where c_D is the drag coefficient of a single element and λ is now the frontal area of roughness per unit plan area *unoccupied* by roughnesses. A second 'closure' equation is needed – Inoue (1963) assumed that the mixing length l is independent of height, where

$$T_{12} = \rho l^2 \frac{du}{dz}\left|\frac{du}{dz}\right|,$$

and thus found

$$u = u_h e^{\alpha(z/h-1)}, \tag{17}$$

where α is the attenuation coefficient

$$\alpha = c_D \lambda h^2/4l^2.$$

This expression cannot be correct near $z = 0$ where u must vanish, but nevertheless it has often been found to be a good approximation to the velocity profile in rod-like canopies. Cionco (1972) has obtained values of α for different roughness types by fitting equation (17) to measured velocity profiles. He found that α lies in the range 1–2 for moderately dense arrays of semi-rigid elements (like mature corn and some types of trees).

Equations (16) and (17) can be used to find T_{12}, and an expression for d is then found by substitution into equation (15) (where the uw term is now zero). The final result is, assuming $\bar{h} \ll h$,

$$\frac{d}{h} = 1-(1-e^{-2\alpha})/2\alpha. \tag{18}$$

This expression is plotted in figure 6. For the typical range $1 < \alpha < 2$, it is seen that the predicted values of d/h agree well with those of table 1.

The solution can be continued to predict z_0 by matching the velocity and shear stress with the logarithmic velocity profile at an appropriate height. Matching at $z = h$ leads to

$$\frac{h-d}{z_0} = \exp\left|\frac{\kappa h}{\alpha l_h}\right|, \tag{19}$$

which cannot be evaluated without a reasonable value for l_h/h. For example if it is assumed that l_h for small values of α retains its smooth-wall value ($l_h = \kappa h$ at $z = h$) then we find

$$\frac{z_0}{h} = \frac{(1-e^{-2\alpha})}{2\alpha e^{1/\alpha}}. \tag{20}$$

This expression is also plotted in figure 6, where again the range $1 < \alpha < 2$ gives good agreement with an average value for vegetation, $z_0/h = 0{\cdot}15$.

6. Conclusions

Once the displacement height d is recognized as the elevation at which the mean drag appears to act on the flow well above the surface roughness elements, it is readily shown to correspond also to the average displacement height of the total shear stress. This result has been shown to be intuitively appealing and to agree with the available experimental results. It has also been demonstrated that d is determined by the distribution of forces on the surface, whereas the roughness length z_0 is determined by the magnitude of these forces.

A simple analytical solution for rod-type roughnesses and a wide range of experimental results both show that the variation of roughness heights about the average elevation is the primary factor on which d depends. In many commonly encountered types of roughness the expression $d = 0{\cdot}7$ times roughness height gives a good estimate. However, it is also clear that d may be strongly dependent on the density of roughness elements.

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